Continuity of Nonlinear Integral Functionals and Existence Theory of Variational Problems(Nonlinear Analysis and Mathematical Economics)

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数理解析研究所講究録
829巻 1993年 192-199

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by

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1. Introduction

An integrand $u:[0,T]\times\mathbb{R}^{\ell}\times\mathbb{R}^{\ell}\longrightarrow\overline{\mathbb{R}}$ is assumed to be given. And let $\mathscr{A}$ be a nonempty subset (called admissible set) of the Sobolev space $\mathfrak{W}^{1,p}([0,T],\mathbb{R}^{\ell})$. The typical problem in the calculus of variations is to minimize (or maximize) the nonlinear integral functional $J:\mathscr{A}\longrightarrow\overline{\mathbb{R}}$ defined by

$$J(x)=\int_0^T u(t,x(t),\dot{x}(t))dt$$

on $\mathscr{A}$.

In order to accomplish the existence proof of variational problems of this kind, it is clearly an indispensable routine task to find out reasonable conditions which assure the continuity of the functional J.

Let $\{x_n\}$ be a minimizing sequence in $\mathscr{A}$ which converges to some function $x^*\in\mathscr{A}$ (in a suitable topology). Then what we actually need to show is the existence of a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that

$$J(x^*)\leq\liminf_{n'} J(x_{n'}),$$

i.e.

$$\int_0^T u(t,x^*(t),\dot{x}^*(t))dt\leq\liminf_{n'}\int_0^T u(t,x_{n'}(t),\dot{x}_{n'}(t))dt.$$

In this paper, we begin by discussing the significant relevance of Ioffe's eminent result for our target, and then proceed to show the way to its extension to the case of a nonlinear integral functional defined on the Sobolev space

$\mathfrak{W}^{1,p}([0,T],\mathfrak{X})$, where $\mathfrak{X}$ is a real Banach space of infinite dimension rather than $\mathbb{R}^{\ell}$.

2. Ioffe's Theorem

Let me start with the comparison of the following two nonlinear integaral functionals, $J:\mathfrak{W}^{1,p}([0,T],\mathbb{R}^{\ell})\longrightarrow\overline{\mathbb{R}}$ and $H:\mathfrak{L}^{p}([0,T],\mathbb{R}^{\ell})\times\mathfrak{L}^{q}([0,T],\mathbb{R}^{\ell})\longrightarrow\overline{\mathbb{R}}$ $(p,q\geq 1)$:

$$J(x)=\int_0^T u(t,x(t),\dot{x}(t))dt,$$

$$H(x,y)=\int_0^T u(t,x(t),y(t))dt,$$

where the integrand $u:[0,T]\times\mathbb{R}^{\ell}\times\mathbb{R}^{\ell}\longrightarrow\mathbb{R}$ is assumed to be given. We say that the functional H is (p,q) *-sequentially lower semi-continuous* if the functional H is sequentially lower semi-continuous with respect to the strong topology for $\mathfrak{L}^{p}([0,T],\mathbb{R}^{\ell})$ and the weak topology for $\mathfrak{L}^{q}([0,T],\mathbb{R}^{\ell})$.

The key point is that we can reduce the continuity-test of J with respect to the weak topology to the (p,q)-continuity-test of H having recourse to the weak convergence thorem in $\mathfrak{W}^{1,p}([0,T],\mathbb{R}^{\ell})$ as follows.

THEOREM 1 *Let $\langle x_n\rangle$ be a sequence in $\mathfrak{W}^{1,p}([0,T],\mathbb{R}^{\ell})$ $(p\geq 1)$ which weakly converges to some $x^*\in\mathfrak{W}^{1,p}$. Then there exists a subsequence $\langle x_{n'}\rangle$ of $\langle x_n\rangle$ such that*

(i) *$\langle x_{n'}\rangle$ uniformly converges to x^*, and*

(ii) *$\langle \dot{x}_{n'}\rangle$ weakly converges to $\dot{x}^*$ in $\mathfrak{L}^{p}([0,T],\mathbb{R}^{\ell})$.*

Assume that H is (p,p)-sequentially lower semi-continuous. And let $\langle x_n\rangle$ be a sequence in $\mathfrak{W}^{1,p}([0,T],\mathbb{R}^{\ell})$ which weakly converges to some $x^*\in\mathfrak{W}^{1,p}$. According to Theorem 1, there exists some subsequence (for the sake of simplicity, we do not change notations) such that

(i) $x_n\longrightarrow x^*$ uniformly, and (ii) $\dot{x}_n\longrightarrow\dot{x}^*$ weakly in $\mathfrak{L}^{p}$.

Hence by the (p,p)-sequential lower semi-continuity of H, we obtain

$$H(x^*,\dot{x}^*) \leq \liminf_n H(x_n,\dot{x}_n),$$

which is equivalent to our desired goal :

$$J(x^*) \leq \liminf_n J(x_n).$$

Thus, in general, the fundamental problem is to find out some conditions which guarantee the (p,q)-lower semi-continuity of H.

DEFINITION Let $(\Omega, \mathcal{E})$ be a measuable space, X a topological space, and $\mathfrak{V}$ a topological vector space. A function $u: \Omega \times X \times \mathfrak{V} \longrightarrow \overline{\mathbb{R}}$ is called a *convex-normal integrand* if the following three conditions are satisfied.

(i) u is $(\mathcal{E} \otimes \mathcal{B} \otimes \mathcal{B}(\mathfrak{V}), \mathcal{B}(\mathbb{R}))$-measuable, where $\mathcal{B}(\cdot)$ denotes the Borel σ-field on a topological space.

(ii) For any fixed $\omega \in \Omega$, the function $(x,v) \longmapsto u(\omega,x,v)$ is lower semi-continuous.

(iii) For any fixed $(\omega,x) \in \Omega \times X$, then function $v \longmapsto u(\omega,x,v)$ is convex.

DEFINITION Let $(\Omega, \mathcal{E}, \mu)$ be a measuare space. A function $f: \Omega \times \mathbb{R}^{\ell} \times \mathbb{R}^{k} \longrightarrow \overline{\mathbb{R}}$ is said to have *the lower compactness property* if $\{f^-(\omega, x_n(\omega), y_n(\omega)) \mid n=1,2,\cdots\cdots\}$ is weakly relatively compact in $\mathcal{L}^1(\Omega, \mathbb{R})$ for any sequence $\{(x_n, y_n)\}$ in $\mathcal{L}^p(\Omega, \mathbb{R}^{\ell}) \times \mathcal{L}^q(\Omega, \mathbb{R}^k)$ such that

a. $\{x_n\}$ is strongly convergent,

b. $\{y_n\}$ is weakly convergent, and

c. $\sup_n \int_\Omega f(\omega, x_n(\omega), y_n(\omega)) d\mu \leq C$ for some $C < \infty$.

(f^- denotes the negative part of f.)

The following theorem due to A.D. Ioffe gives a definitive criterion of the (p,q)-sequential lower semi-continuity of H.

THEOREM 2 (Ioffe [5]) *Assume that $(\Omega, \mathcal{E}, \mu)$ is a non-atomic complete finite* measure space, and $u:\Omega\times\mathbb{R}^{\ell}\times\mathbb{R}^{k}\longrightarrow\overline{\mathbb{R}}$ *is a proper convex normal integrand. The nonlinear integral functional* $H:\mathcal{L}^{p}(\Omega,\mathbb{R}^{\ell})\times\mathcal{L}^{q}(\Omega,\mathbb{R}^{k})\longrightarrow\overline{\mathbb{R}}$ $(1\leq p,q<+\infty)$ *is defined by*

$$H(x,y)=\int_{\Omega}u(\omega,x(\omega),y(\omega))d\mu.$$

We also assume that $H(x,y)<+\infty$ *for some* $(x,y)\in\mathcal{L}^{p}\times\mathcal{L}^{q}$. *Then the following two statements are equivalent.*

(i) H *is* (p,q) *-sequentially lower semi-continuous, and* $H(x,y)>-\infty$ *for all* $(x,y)\in\mathcal{L}^{p}\times\mathcal{L}^{q}$.

(ii) *The integrand* u *has the lower compactness property.*

COROLLARY 1 *Assume that* $(\Omega,\mathcal{E},\mu)$ *is a non-atomic complete finite measure space and* $u:\Omega\times\mathbb{R}^{\ell}\times\mathbb{R}^{k}\longrightarrow\overline{\mathbb{R}}$ *is a proper convex normal integrand.* $(1\leq p,q<+\infty.)$

(i) *If* u *is non-negative, then* H *is* (p,q) *-sequentially lower semi-continuous.*

(ii) *If there exist some* $a\in\mathcal{L}^{q'}(\Omega,\mathbb{R}^{k})$ (*where* $1/q+1/q'=1$) *and* $b\in\mathcal{L}^{1}(\Omega,\mathbb{R})$ *such that*

$$u(\omega,x,y)\geq\langle a(\omega),y\rangle+b(\omega)\quad \textit{for all } (\omega,x,y)\in\Omega\times\mathbb{R}^{\ell}\times\mathbb{R}^{k},$$

then H *is* (p,q) *-sequentially lower semi-continuous.*

(iii) *If there exist some* $c\in\mathbb{R}$ *and* $b\in\mathcal{L}^{1}(\Omega,\mathbb{R})$ *such that*

$$u(\omega,x,y)\geq-c(\|x\|+\|y\|)+b(\omega),$$

then H *is* $(1,1)$ *-sequentially lower semi-continuous.*

3. Weak Convergence in $\mathfrak{W}^{1,p}([0,T],\mathfrak{X})$

Let $\mathfrak{X}$ be a real Banach apace and an integrand $u:[0,T]\times\mathfrak{X}\times\mathfrak{X}\longrightarrow\overline{\mathbb{R}}$ be given. And consider the nonlinear integral functional $J:\mathfrak{W}^{1,p}([0,T],\mathfrak{X})\longrightarrow\overline{\mathbb{R}}$ defined by

$$J(x)=\int_0^T u(t,x(t),\dot{x}(t))dt.$$

It seems a natural way of reasoning to reduce the continuity-test of J to that of another integral functional $H: \mathcal{L}^p([0,T],\mathfrak{X})\times\mathcal{L}^q([0,T],\mathfrak{X})\longrightarrow\overline{\mathbb{R}}$ defined by

$$H(x,y)=\int_0^T u(t,x(t),y(t))dt$$

as in the case of $\mathfrak{X}=\mathbb{R}^\ell$.

However we should remind of the fact that unfortunately the analog of Theorem 1 does not hold if we replace $\mathbb{R}^\ell$ by a Banach space of infinite demension.

COUNTER-EXAMPLE (Cecconi [4]) Let $\mathcal{H}$ be a separable Hilbert space and $\{\varphi_n\}$ $(n=1,2,\cdots\cdots)$ a complete orthonormal system in $\mathcal{H}$. Furthermore define a sequence $\{x_n:[0,1]\longrightarrow\mathcal{H}\}$ of functions by $x_n(t)=t\varphi_n$ $(n=1,2,\cdots\cdots)$. Then it can easily be checked that $\{x_n\}$ converges weakly to $x^*\equiv 0$ in $\mathfrak{W}^{1,1}([0,1],\mathcal{H})$. However there exists no subsequence of $\{x_n\}$ which strongly converges to x^* in $\mathcal{L}^1([0,T],\mathcal{H})$. (For a systematic exposition about the theory of the Sobolev space consisting of vector-valued functions, see Maruyama [9].)

In order to overcome this difficulty, we need a new weak convergence theorem for $\mathfrak{W}^{1,p}([0,1],\mathfrak{X})$ as a substitute for Theorem 1. We denote by $\mathcal{H}_w$ the space $\mathcal{H}$ endowed with the weak topology.

THEOREM 3 (Maruyama [6]) *Let $\mathcal{H}$ be a real separable Hilbert space and consider a sequence $\{x_n\}$ in the Sobolev space $\mathfrak{W}^{1,p}([0,1],\mathcal{H})$. Assume that*

(i) *the set $\{x_n(t)\}_{n=1}^\infty$ is bounded in $\mathcal{H}$ for each $t\in[0,T]$, and*

(ii) *there exists some $\psi\in\mathcal{L}^p([0,T],(0,+\infty))$ such that*

$$\|\dot{x}_n(t)\|\leq\psi(t)\quad\text{a.e.}$$

Then there exist a subsequence $\{z_n\}$ of $\{x_n\}$ and some $x^*\in\mathfrak{W}^{1,p}([0,1],\mathcal{H})$ such that

(a) $z_n\longrightarrow x^*$ *uniformly in $\mathcal{H}_w$ on $[0,T]$, and*

(b) $\dot{z}_n\longrightarrow\dot{x}^*$ *weakly in $\mathcal{L}^p([0,T],\mathcal{H})$.*

4. Integral Functionals on the Space of Bochner Integrable Functions

We now redefine the concept of the lower compactness property in a new context.

DEFINITION Let $(\Omega, \mathcal{E}, \mu)$ be a measure space, X a topological space, and $\mathfrak{V}$ a Banach space. We shall denote by $\mathfrak{M}(\Omega, X)$ the space of all the $(\mathcal{E}, \mathcal{B}(X))$-measureable mappings of Ω into X. Then a function $f: \Omega \times X \times \mathfrak{V} \longrightarrow \overline{\mathbb{R}}$ is said to have the *lower compactness property* if $\{f^-(\omega, x_n(\omega), v_n(\omega))\}$ is weakly relatively compact in $\mathcal{L}^1(\Omega, \mathbb{R})$ for any sequence $\{(x_n, v_n)\}$ in $\mathfrak{M}(\Omega, X) \times \mathcal{L}^p(\Omega, \mathfrak{V})$ $(1 \leq p < +\infty)$ which satisfies the following three conditions :

a. $x_n(\omega) \longrightarrow x^*(\omega)$ pointwise as $n \to \infty$ for some $x^* \in \mathfrak{M}(\Omega, X)$,

b. $v_n \longrightarrow v^*$ weakly in $\mathcal{L}^p(\Omega, \mathfrak{V})$ as $n \to \infty$ for some $v^* \in \mathcal{L}^p(\Omega, \mathfrak{V})$, and

c. $\sup_n \int_\Omega f(\omega, x_n(\omega), v_n(\omega)) d\mu \leq C$ for some $C < +\infty$.

THEOREM 4 (Maruyama [7]) *Let* $(\Omega, \mathcal{E}, \mu)$ *be a compact space with a positive Radon measure* μ, X *a Souslin topological space, and* C *a bounded disked subset of a separable reflexive Banach space* $\mathfrak{V}$. *Let* $f: \Omega \times X \times C \longrightarrow \overline{\mathbb{R}}$ *be a proper convex normal integrand with the lower compactness property.*

Futhermore let $\{x_n : \Omega \longrightarrow X\}$ *be a sequence in* $\mathfrak{M}(\Omega, X)$ *which converges pointwise to some mapping* $x^* \in \mathfrak{M}(\Omega, X)$; i.e.

$$x_n(\omega) \longrightarrow x^*(\omega) \quad \text{for each } \omega \in \Omega.$$

Let $\{v_n : \Omega \longrightarrow C\}$ *be a sequence in* $\mathcal{L}^p(\Omega, C)$ $(1 \leq p < +\infty)$ *which weakly converges to some* $v^* \in \mathcal{L}^p(\Omega, C)$; i.e.

$$\int_\Omega \langle \eta(\omega), v_n(\omega) \rangle d\mu \longrightarrow \int_\Omega \langle \eta(\omega), v^*(\omega) \rangle d\mu$$

for every $\eta \in \mathcal{L}^q(\Omega, \mathfrak{V}')$ $(1/p + 1/q = 1)$.

(Remark : Here we should remind of the fact that $\mathfrak{V}$ *has the Radon-Nikodym property.)*

Finally assume that there exists at least one $\overline{v} \in \mathcal{L}^p(\Omega, C)$ *such that*

$$\int_\Omega | f(\omega, x^*(\omega), \overline{v}(\omega)) | d\mu < +\infty.$$

Then it follows that

$$\lim_n \inf \int_\Omega f(\omega, x_n(\omega), v_n(\omega))d\mu \geq \int_\Omega f(\omega, x^*(\omega), v^*(\omega))d\mu.$$

Taking account of the fact that the bounded closed convex subset of a separable Hilbert space endowed with the weak topology is a Polish space (and hence Souslin space), we immediately obtain the following corollary.

COROLLARY 2 *Let* $\mathbb{H}$ be a real separable Hilbert space and $u:[0,T]\times\mathbb{H}\times\mathbb{H} \longrightarrow \overline{\mathbb{R}}$ a proper convex normal integrand with the lower compactness property. And let $\mathcal{A}$ *be a subset of* $\mathbb{W}^{1,p}([0,T],\mathbb{H})$ *such that both of* $\{\dot{x}(t) \mid x\in\mathcal{A}, t\in[0,T]\}$ *and* $\{x(t) \mid x\in\mathcal{A}, t\in[0,T]\}$ *are bouded in* $\mathbb{H}$. *The nonlinear integral functional* $J:\mathcal{A}\longrightarrow\overline{\mathbb{R}}$ *is defined by*

$$J(x)=\int_0^T u(t,x(t),\dot{x}(t))dt \ ; \ x\in\mathcal{A}.$$

Then any sequence $\{x_n\}$ *in* $\mathcal{A}$ *which weakly converges to some* $x^*\in\mathcal{A}$ *has a subsequence* $\{x_{n'}\}$ *such that*

$$J(x^*)\leq \lim_n \inf J(x_{n'}).$$

An applicatin of this result to the existence proof for a variational problem governed by a differential inclusion can be found in Maruyama [8].

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